

Flats of a positroid

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Abstract

A positroid is a special case of a realizable matroid, that arose from the study of totally non-negative part of the Grassmannian by Postnikov [11]. Postnikov demonstrated that positroids are in bijection with certain interesting classes of combinatorial objects, such as Grassmann necklaces and decorated permutations. Oh showed that the bases of a positroid can be described nicely in terms of the Grassmann necklace and decorated permutations [9]. In this paper, we show how to describe the flats of a positroid. In particular, we show how to describe the inseparable flats and their rank using the associated decorated permutation.

1 Introduction

A matrix is totally positive (respectively totally nonnegative) if all its minors are positive (respectively nonnegative) real numbers. These matrices have a number of remarkable properties: for example, an $n \times n$ totally positive matrix has n distinct positive eigenvalues. The space of these matrices can be grouped up into topological cells, with each cell completely parametrized by a certain planar network [4]. The idea of total positivity found numerous applications and was studied from many different angles, including oscillations in mechanical systems, stochastic processes and approximation theory, and planar resistor networks [4].

Now, instead of considering $n \times n$ matrices with nonnegative minors, consider a full-rank $k \times n$ matrix with all maximal minors nonnegative. This arose from the study of the totally nonnegative part of the Grassmannian by Postnikov [11]. The set of nonzero maximal minors of such matrices forms a positroid, which is a matroid used to encode the topological cells inside the nonnegative part of the Grassmannian. Positroids have a number of nice combinatorial properties. In particular, Postnikov demonstrated that positroids are in bijection with certain interesting classes of combinatorial objects, such as Grassmann necklaces and decorated permutations. Recently, positroids have seen increased applications in physics, with use in the study of scattering amplitudes [2] and the study of shallow water waves [7].

A matroid can be described in multiple ways, using bases, independent sets, circuits, rank function, flats, etc. There have been multiple results on the bases of a positroid: The set of bases can be described nicely from the Grassmann necklace [9], and the polytope coming from the bases can be described using the cyclic intervals [8],[1]. In this paper we shift the focus to the flats and the rank function of a positroid. We provide an algorithm for computing the rank of an arbitrary set in a positroid. Using this, we will describe the inseparable flats of a positroid using the associated decorated permutation. As a by-product, we will obtain a description of the facets of the independent set polytope of a positroid.

The structure of the paper is as follows. In Section 2, we go over the background materials needed for this paper, including the basics of matroids, positroids, Grassmann necklace and decorated permutations. In section 3 we show a basis-exchange like property for cyclic intervals that

works for positroids. In section 4, we describe which intervals are flats using the associated decorated permutation. In section 5, we present an algorithm for computing the rank of an arbitrary set in a positroid. In section 6, we show how to describe the flats of a positroid using the associated decorated permutation.

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2 Background materials

2.1 Matroids

In this section we review the basics of matroids that we will need. We refer the reader to [10] for a more in-depth introduction to matroid theory.

Definition 1. A **matroid** is a pair (E, \mathcal{B}) consisting of a finite set E , called the **ground set** of the matroid, and a nonempty collection of subsets $\mathcal{B} = \mathcal{B}(\mathcal{M})$ of E , called the **bases** of \mathcal{M} , which satisfy the **basis exchange axiom**:

If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, then there exists $b_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

A subset $F \subset E$ is called **independent** if it is contained in some basis. All maximal independent sets contained in a given set $A \subset E$ have the same size, called the **rank** $rk(A)$ of A . The rank of the matroid \mathcal{M} , denoted as $rk(\mathcal{M})$, is given by $rk(E)$. The **closure** of a set A is denoted as \bar{A} , and stands for the biggest set that contains A and has the same rank. A set is a **flat** if its closure is same as itself. A set E is called **separable** in a matroid if one can partition E into E_1 and E_2 such that $rk(E) = rk(E_1) + rk(E_2)$. An element $e \in E$ is a **loop** if it is not contained in any basis. An element $e \in E$ is a **coloop** if it is contained in all bases. A matroid \mathcal{M} is **loopless** if it does not contain any loops. The **dual** of \mathcal{M} is a matroid $\mathcal{M}^* = (E, \mathcal{B}')$ where $\mathcal{B}' = \{E \setminus B \mid B \in \mathcal{B}(\mathcal{M})\}$. By using the basis exchange axiom on the dual matroid, we get the following **dual basis exchange axiom**:

If $B_1, B_2 \in \mathcal{B}$ and $b_2 \in B_2 \setminus B_1$, then there exists $b_1 \in B_1 \setminus B_2$ such that $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

The following property of the rank function will be crucial for proving our results:

Theorem 1. [10] The rank function is semimodular, meaning that $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$ for any subset A and B of E .

We now go over polytopes related to matroids.

Definition 2. Given a matroid $\mathcal{M} = ([n], \mathcal{B})$, the (basis) matroid polytope $\Gamma_{\mathcal{M}}$ of \mathcal{M} is the convex hull of the indicator vectors of the bases of \mathcal{M} :

$$\Gamma_{\mathcal{M}} = \text{convex}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n,$$

where $e_B := \sum_{i \in B} e_i$ and $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{R}^n .

Definition 3. Given a matroid $\mathcal{M} = ([n], \mathcal{B})$, the independent set polytope $P_{\mathcal{M}}$ of \mathcal{M} is the convex hull of the indicator vectors of the independent sets of \mathcal{M} :

$$P_{\mathcal{M}} = \text{convex}\{e_I | I \subset B \in \mathcal{B}\} \subset \mathbb{R}^n,$$

where $e_I := \sum_{i \in I} e_i$ and $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{R}^n .

There is a nice description for the facets of an independent set polytope.

Theorem 2 (Theorem 40.5. of [12]). *If \mathcal{M} is loopless, the following is a minimal system for the independent set polytope of \mathcal{M} :*

- $x_e \geq 0, e \in E,$
- $x_F := \sum_{e \in F} x_e \leq rk(F), F \text{ is a nonempty inseparable flat of } \mathcal{M},$

One of the main result of this paper will be describing all the inseparable flats and their rank of a given positroid.

2.2 Positroids

In this section we go over the basics of positroids. Positroids were originally defined in [11] as the column sets coming from nonzero maximal minors in a totally nonnegative matrix (a matrix such that all maximal minors are nonnegative). For example, consider the following matrix:

$$A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 0 \end{pmatrix}$$

The nonzero maximal minors comes from column sets $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}$. This collection forms a positroid. However in this paper, we will use an equivalent definition using Grassmann necklace and Gale orderings.

Definition 4. Let $d \leq n$ be positive integers. A **Grassmann necklace** of type (d, n) is a sequence (I_1, \dots, I_n) of d -subsets $I_k \in \binom{[n]}{d}$ such that for any $i \in [n]$,

- if $i \in I_i$ then $I_{i+1} = I_i \setminus \{i\} \cup \{j\}$ for some $j \in [n]$,
- if $i \notin I_i$ then $I_{i+1} = I_i$,

where $I_{n+1} = I_1$.

The **cyclically shifted order** $<_i$ on the set $[n]$ is the total order

$$i <_i i + 1 <_i \dots <_i n <_i 1 <_i \dots <_i i - 1.$$

For any rank d matroid \mathcal{M} with ground set $[n]$, let I_k be the lexicographically minimal basis of \mathcal{M} with respect to $<_k$, and denote

$$\mathcal{I}(\mathcal{M}) := (I_1, \dots, I_n),$$

which forms a Grassmann necklace [11].

The **Gale order** on $\binom{[n]}{d}$ (with respect to $<_i$) is the partial order $<_i$ defined as follows: for any two d -subsets $S = \{s_1 <_i \dots <_i s_d\}$ and $T = \{t_1 <_i \dots <_i t_d\}$ of $[n]$, we have $S \leq_i T$ if and only if $s_j \leq_i t_j$ for all $j \in [d]$ [5].

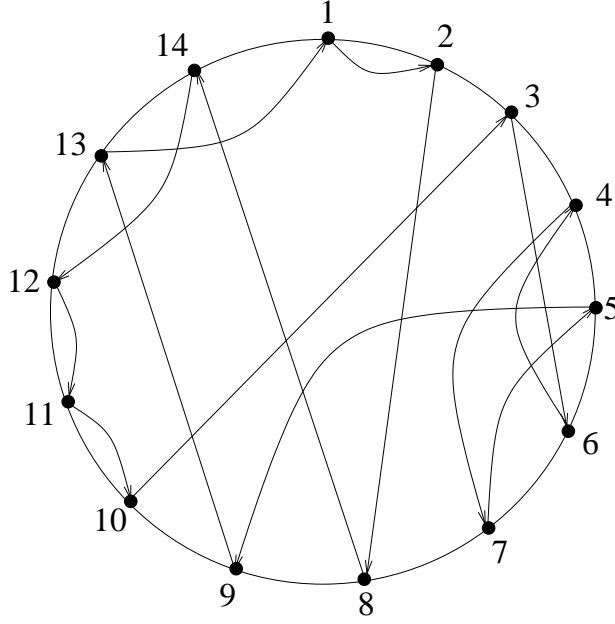


Figure 1: A decorated permutation.

Theorem 3 ([11],[9]). *Let $\mathcal{I} = (I_1, \dots, I_n)$ be a Grassmann necklace of type (d, n) . Then the collection*

$$\mathcal{B}(\mathcal{I}) := \{B \in \binom{[n]}{d} \mid B \geq_j I_j, \text{ for all } j \in [n]\}$$

is the collection of bases of a rank d positroid $\mathcal{M}(\mathcal{I}) := ([n], \mathcal{B}(\mathcal{I}))$. Moreover, for any positroid \mathcal{M} , we have $\mathcal{M}(\mathcal{I}(\mathcal{M})) = \mathcal{M}$.

It is worth noting that the class of positroids is closed under taking restriction, contraction and the dual [1].

Definition 5. *A decorated permutation of the set $[n]$ is a bijection π of $[n]$ whose fixed points are colored either white or black. A weak i -exceedance of a decorated permutation π is an element $j \in [n]$ such that either $j <_i \pi(j)$ or j is a fixed point colored black.*

Given a Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ we can construct a decorated permutation $\pi_{\mathcal{I}}$ of the set $[n]$ in the following way.

- If $I_{i+1} = I_i \setminus \{i\} \cup \{j\}$ for $i \neq j$ then $\pi_{\mathcal{I}}(j) := i$.
- If $I_{i+1} = I_i$ and $i \notin I_i$ then i is a fixed point colored white.
- If $I_{i+1} = I_i$ and $i \in I_i$ then i is a fixed point colored black.

Conversely, given a decorated permutation π of $[n]$ we can construct a Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ by letting I_k be the set of weak k -exceedances of π . This gives a bijection between the Grassmann necklace and decorated permutations [11].

For example, take a look at the decorated permutation (since it has no fixed points, it is the usual permutation) in Figure 1. It is the permutation $[2, 8, 6, 7, 9, 4, 5, 14, 13, 3, 10, 11, 1, 12]$

under the usual bracket notation. The weak 1-exceedances of the permutation is given by the set $\{1, 3, 4, 5, 10, 11, 12\}$, and this is I_1 of the associated Grassmann necklace.

Given $a, b \in [n]$, we define the (cyclic) interval $[a, b]$ to be the set $\{x | x \leq_a b\}$. These cyclic intervals plays an important role in the structure of a positroid [6].

Theorem 4 ([8],[1]). *A matroid \mathcal{M} of rank d on $[n]$ is a positroid if and only if its matroid polytope $\Gamma_{\mathcal{M}}$ can be described by the equality $x_1 + \dots + x_n = d$ and inequalities of form*

$$\sum_{l \in [a, b]} x_l \leq rk([a, b]), \text{ with } i, j \in [n].$$

We will later show an analogous result for independent set polytopes of positroids.

Remark 1. *If a positroid \mathcal{M} has loops or coloops, it is enough to study the positroid \mathcal{M}' obtained by deleting the loops and the coloops to study the structural properties of \mathcal{M} . So throughout this paper, we will assume that our positroid has neither loops nor coloops. This means that the associated decorated permutation has no fixed points.*

3 Interval exchange property

In this section we develop a basis exchange technique for positroids that will serve as a powerful tool throughout the paper.

The following Lemma follows from Theorem 3.1. of [3].

Lemma 1. *Take S to be an arbitrary subset of $[n]$. Pick an arbitrary subset $I \in \binom{[n]}{k}$, where $k < |S|$. There exists $I' \in \binom{S}{k}$ such that $\{J | J \leq I, J \in \binom{S}{k}\} = \{J | J \leq I', J \in \binom{S}{k}\}$.*

Another way to state the above lemma is that the contraction of a **Schubert matroid**, a matroid of form $\{J | J \geq I\}$ for some subset I of $[n]$, is again a Schubert matroid.

The following property follows from the definition of Grassmann necklaces and the proof will be omitted.

Lemma 2 (Sharing property). *Let a and b be arbitrary elements of $[n]$. Then we have $I_a \cap [b, a) \subset I_b$.*

We begin our analysis of the cyclic intervals of a positroid.

Lemma 3. *The maximal number of elements a basis of \mathcal{M} can have in an interval $[a, b]$ is given by $|I_a \cap [a, b]|$. Similarly, the minimal number of elements a basis of \mathcal{M} can have in an interval (b, a) is given by $|I_a \cap (b, a)|$.*

Proof. The claim follows from the bound $I_a \leq_a B$ and the fact that I_a is a basis of \mathcal{M} . □

The above lemma suggests that given a cyclic interval $[a, b]$, the set $I_a \cap [a, b]$ plays a crucial role in studying that interval. Next we present the main result of this section:

Proposition 1 (Interval exchange property of positroids). *Pick any basis $J \in \mathcal{M}$ and an arbitrary cyclic interval $[a, b]$. There exists $J' \in \mathcal{M}$ that is obtained by replacing $J \cap [a, b]$ with a subset of $I_a \cap [a, b]$.*

Proof. Denote A as $J \cap [a, b]$. Using Lemma 1, we get $A' \in \binom{I_a \cap [a, b]}{|A|}$ such that for any $B \in \binom{I_a \cap [a, b]}{|A|}$ with $B \leq_a A$, we have $B \leq_a A'$. Write J' as the set obtained from J by replacing A with A' . Now we want to show that $J' \in \mathcal{M}$.

Choose q to be an arbitrary element of (b, a) . Let I'_q be the subset of I_q that gets compared with $A = J \cap [a, b]$ in $I_q \leq_q J$. We have $I'_q \leq_q A$. From the way A' was chosen, we have $I'_q \leq_q A' = J' \cap [a, b]$. Combined with the fact that $J \cap (b, a) = J' \cap (b, a)$, we end up with $I_q \leq_q J'$.

Now choose q to be an arbitrary element of $[a, b]$. For any set B with cardinality $|J|$, using the ordering \leq_q , denote $\text{tail}(B)$ as the last $|J' \cap [a, q]|$ elements of B , $\text{head}(B)$ as the first $|J' \cap [q, b]|$ elements of B and $\text{body}(B)$ as the rest. In order to show $I_q \leq_q J'$, it is enough to show that $\text{head}(I_q) \leq_q \text{head}(J')$, $\text{body}(I_q) \leq_q \text{body}(J')$ and $\text{tail}(I_q) \leq_q \text{tail}(J')$. First for the head part, since $J' \cap [q, b] \subset I_a \cap [q, b] \subset I_q \cap [q, b]$ (the first containment follows from the construction of J' , and the second containment follows from Lemma 2, the sharing property), we get $\text{head}(I_q) \leq_q \text{head}(J')$. For the body part, since $J' \cap (b, a) = J \cap (b, a)$ and $|J' \cap [b, q]| \leq |J \cap [b, q]|$, we get $\text{body}(J') \geq_q \text{body}(J) \geq_q \text{body}(I_q)$. Lastly for the tail part, the part of I_q that gets compared with $J \cap [a, q]$ in $I_q \leq_q J$ is smaller than the part obtained from J' by taking the biggest $|A \cap [a, q]|$ elements under \leq_q (Due to $I_q \cap [a, q] \subset I_a$ and the way A' was chosen). This implies $\text{tail}(I_q) \leq_q \text{tail}(J')$.

Hence we get $I_q \leq_q J'$ for all $q \in [n]$, which allows us to conclude that $J' \in \mathcal{M}$. \square

Corollary 1. *Let J be a basis of \mathcal{M} . If $|J \cap [a, b]|$ is maximal among all $|B \cap [a, b]|, B \in \mathcal{M}$, we may replace $J \cap [a, b]$ with $I_a \cap [a, b]$ to get another basis J' of \mathcal{M} . If $|J \cap (b, a)|$ is minimal among all $|B \cap (b, a)|, B \in \mathcal{M}$, we may replace $J \cap (b, a)$ with $I_a \cap (b, a)$ to get another basis J' of \mathcal{M} .*

Proof. The first statement follows from the proposition above. The second statement follows from the first by applying it on the dual poset. \square

4 Describing interval flats

In this section we come up with a criterion for checking if an interval is a flat or not. We will call a cyclic interval an **interval flat** if it is a flat. In the lemma below, we describe exactly when one can replace an element of I_a with a desired element outside I_a , to get another basis in the poset.

Lemma 4. *Let I_a be an element of the Grassmann necklace of \mathcal{M} and let y be an arbitrary element outside I_a in $[n]$. If $I_a \cap [a, y] \subset I_y$, then there does not exist x such that $I_a \setminus \{x\} \cup \{y\} \in \mathcal{M}$. Otherwise, pick x to be the rightmost (biggest under \leq_a) among elements of $(I_a \setminus I_y) \cap [a, y]$. Then $I_a \setminus \{x\} \cup \{y\} \in \mathcal{M}$.*

Proof. We will use I' to denote $I_a \setminus \{x\} \cup \{y\} \in \mathcal{M}$. For the first statement, if $I_a \cap [a, y] \subset I_y$, then I_a already has the minimal possible number of elements in the interval $[a, y]$. Hence there cannot exist x such that $I' \in \mathcal{M}$. For the second statement, we need to show that $I_q \leq_q I'$ for all $q \in [n]$. For $q \in (y, x]$, we have $I' \geq_q I_a \geq_q I_q$ since $x <_q y$. For $q \in (x, y]$, we have $I' \geq_q I_y \geq_q I_q$ since $I' \cap (x, a) \subset I_y$. \square

In the next lemma, we show that the behavior of $I_a \cap [a, b]$ is enough to determine whether $[a, b]$ is a flat or not.

Lemma 5. *Let $[a, b]$ be a cyclic interval of $[n]$. This interval is a flat if and only if for any $y \notin [a, b]$, there exists some $x \notin [a, b]$ such that $I_a \setminus \{x\} \cup \{y\} \in \mathcal{M}$.*

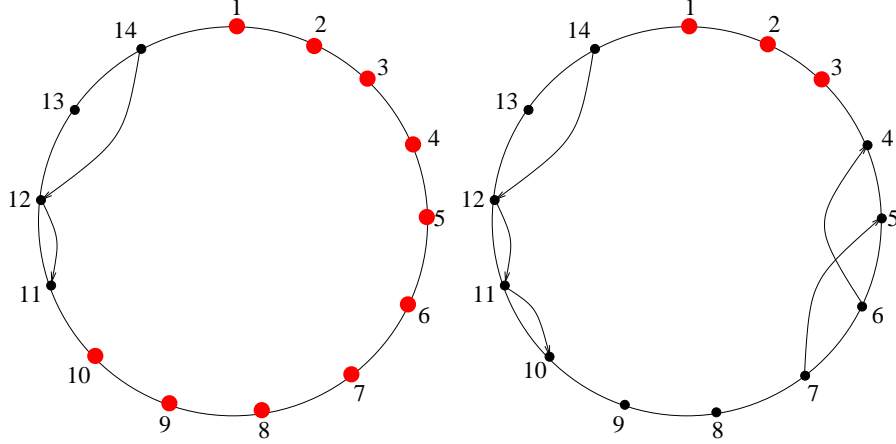


Figure 2: The interval $[1, 10]$ is a flat. The interval $[1, 3]$ is not.

Proof. From the definition of flats and Lemma 3, the cyclic interval $[a, b]$ is a flat if and only if for any $y \notin [a, b]$, there exists a basis $J \in \mathcal{M}$ such that $y \in J$ and $|J \cap [a, b]| = |I_a \cap [a, b]|$. One direction is obvious : If for any $y \notin [a, b]$, there exists some $x \notin [a, b]$ such that $I_a \setminus \{x\} \cup \{y\} \in \mathcal{M}$, then $[a, b]$ is a flat.

We have to show the other direction. Suppose $[a, b]$ is a flat and for any $y \notin [a, b]$, we have a basis $J \in \mathcal{M}$ such that $y \in J$ and $|J \cap [a, b]| = |I_a \cap [a, b]|$. From Proposition 1, the interval exchange property, we may assume that $J \cap [a, b] = I_a \cap [a, b]$. Then we can find some $x \notin [a, b]$ such that $I_a \setminus \{x\} \cup \{y\} \in \mathcal{M}$ holds, from the dual basis exchange property between I_a and J . \square

We now show how to describe the interval flats (cyclic intervals that are flats) directly from the decorated permutation.

Theorem 5. *Let $[a, b]$ be a cyclic interval of $[n]$. This interval is a flat if and only if (b, a) is completely covered by cyclic intervals of form $[x, \pi^{-1}(x)]$ that do not intersect $[a, b]$.*

Proof. From Lemma 4 and Lemma 5, the interval $[a, b]$ is a flat if and only if for any $y \in (b, a)$, there exists some $x \in (I_a \setminus I_y) \cap (b, y)$. Recall that $x \in I_a$ if and only if $x <_a \pi^{-1}(x)$. So the existence of $x \in (I_a \setminus I_y) \cap (b, y)$ is equivalent to the existence of an interval $[x, \pi^{-1}(x)]$ that contains y , and is contained in (b, a) . Therefore, the interval $[a, b]$ is a flat if and only if all elements of (b, a) are covered by such intervals coming from the associated decorated permutation. \square

For example, take a look at Figure 2. The complement of the interval $[1, 10]$ is covered by intervals of form $[x, \pi^{-1}(x)]$ disjoint from $[1, 10]$. So this is a flat. On the other hand, the complement of the interval $[1, 3]$, the elements 8 and 9 in particular, is not covered by intervals of form $[x, \pi^{-1}(x)]$ disjoint from $[1, 3]$. So $[1, 3]$ is not a flat, and its closure is $[1, 3] \cup [8, 9]$.

Remark 2. *The study of cyclic intervals that are flats was motivated from the essential intervals studied in [6]. We would like to point out that the set of essential intervals and the set of interval flats are incomparable: there are essential intervals that are not flats and there are interval flats that are not essential.*

We can show that an arbitrary intersection of interval flats can be described using a similar criterion.

Corollary 2. *Let E be an arbitrary subset of $[n]$. Then E is the intersection of interval flats if and only if the complement is covered by intervals of form $[x, \pi^{-1}(x)]$ that does not intersect E .*

Proof. Let E be an intersection of interval flats. Since the intersection of flats is again a flat [10], we may assume that $E = [a_1, b_1] \cup \dots \cup [a_k, b_k]$ where the $[a_i, b_i]$'s are pairwise disjoint cyclic intervals, the endpoints $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ are cyclically ordered, and E is the intersection of $[a_1, b_k], [a_2, b_1], \dots, [a_k, b_{k-1}]$ where each one of them are interval flats. By Theorem 5, each component of E^c is covered by intervals of form $[x, \pi^{-1}(x)]$ that lie inside them.

For the other direction, write $E = [a_1, b_1] \cup \dots \cup [a_k, b_k]$ where $[a_i, b_i]$'s are pairwise disjoint cyclic intervals and the endpoints $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ are cyclically ordered. Again using Theorem 5, the cyclic intervals $[a_1, b_k], [a_2, b_1], \dots, [a_k, b_{k-1}]$ are all flats. \square

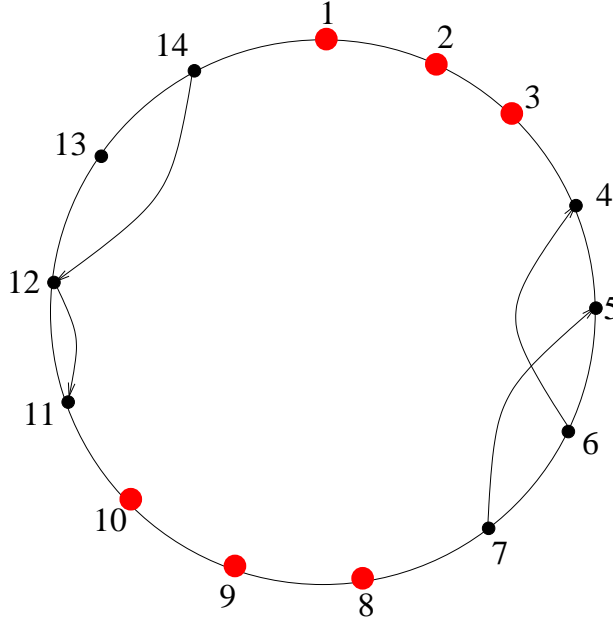


Figure 3: The set $[1, 3] \cup [8, 10]$ is the intersection of flats $[1, 10]$ and $[8, 3]$.

For example, take a look at Figure 3. The complement of $[1, 3] \cup [8, 10]$ consists of the intervals $(3, 8)$ and $(10, 1)$. And each of those intervals is covered by intervals of form $[x, \pi^{-1}(x)]$ that does not intersect $[1, 3] \cup [8, 10]$. So $[1, 3] \cup [8, 10]$ is the intersection of interval flats. In particular, it is the intersection of $[1, 10]$ and $[8, 3]$, both of which are flats.

5 Rank of arbitrary sets

Let $E = [a_1, b_1] \cup \dots \cup [a_k, b_k]$ be the disjoint union of k cyclic intervals, where $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ are cyclically ordered. The rank of E is bounded above by $rk(\mathcal{M})$ minus the sum of the minimal number of elements that a basis of \mathcal{M} can possibly have in each cyclic intervals of the complement of E . So we get $rk(E) \leq rk(\mathcal{M}) - \sum_i \text{minelts}(b_i, a_{i+1})$, where $\text{minelts}(b, a)$ stands for the minimal number of elements that a basis of \mathcal{M} can have in the interval (b, a) (Here the indices of $[k]$ are considered cyclically, so $a_{k+1} = a_1$). Notice that $\text{minelts}(b, a) = rk(\mathcal{M}) - rk([a, b])$. We call this bound the **natural rank bound of E** : $nbd(E) := rk(\mathcal{M}) - \sum_i (rk(\mathcal{M}) - rk([a_i, b_{i-1}]))$.

Definition 6. Let Π be a partition $[k] = T_1 \sqcup \dots \sqcup T_r$ of $[k]$ into pairwise disjoint non-empty subsets. We say that Π is a **non-crossing partition** if there are no $a < b < c < d$ such that $a, c \in T_i$ and $b, d \in T_j$ for some $i \neq j$.

Let Π be an arbitrary non-crossing partition of $[k]$ with T_1, \dots, T_r as its parts. We define $E|_{T_i}$ as the subset of E obtained by taking only the intervals indexed by elements of T_i . By submodularity of the rank function, we get another upper bound on the rank of E : $rk(E) \leq nbd(E, \Pi) := nbd(E|_{T_1}) + \dots + nbd(E|_{T_r})$. So for each non-crossing partition of $[k]$, we get an upper bound on the rank of E . We show that one of those bounds has to be tight in the theorem below.

Before we state the theorem, we prove a lemma that will be useful:

Lemma 6. Pick a, b, c to be arbitrary cyclically ordered elements of $[n]$. Let $J \in \mathcal{M}$ be a basis such that $I_c \cap (b, c) \subset J \cap (b, c)$ and $J \cap [c, a) \subset I_c \cap [c, a)$. Choose x to be the rightmost element of (b, c) (biggest under \leq_a) among $J \setminus I_c$, and choose y to be the leftmost element of $[c, a)$ (smallest under \leq_a) among $I_c \setminus J$. Then $J \setminus \{x\} \cup \{y\} \in \mathcal{M}$.

Proof. For $q \in (y, x]$, we have $J' \geq_q J \geq_q I_q$. For $q \in (x, y]$, we have $J' \geq_q I_c \geq_q I_q$ since $J' \cap (x, y] = I_c \cap (x, y]$ and $J' \geq_c I_c$. Combining these two facts, we get $J' \in \mathcal{M}$. \square

Now we show that one of the upper bounds coming from non-crossing partitions of the intervals of E actually equals the rank of E . We devise a procedure that modifies I_{a_1} and comes up with a basis H , such that $|H \cap E|$ equals $nbd(E, \Pi)$ for some non-crossing partition Π of $[k]$.

Set H^1 as I_{a_1} , the set we are starting with. Now repeat the following procedure, increasing t each time until it reaches k : given H^t , Lemma 6 (using $a = a_1, b = b_t, c = a_{t+1}$) tells us that we can push the elements of $(H^t \setminus I_{a_{t+1}}) \cap (b_t, a_{t+1})$ to elements of $(I_{a_{t+1}} \setminus H^t) \cap [a_{t+1}, a_1)$. Label the resulting basis as H^{t+1} , and it has the property that $H^{t+1} \cap (b_{t+1}, a_{t+2}) \supset I_{a_{t+2}} \cap (b_{t+1}, a_{t+2})$ and $H^{t+1} \cap [a_{t+2}, a_1) \subset I_{a_{t+2}} \cap [a_{t+2}, a_1)$ by the sharing property. Now H^{t+1} satisfies at least one of the following:

- either we ran out of elements to push in (b_t, a_{t+1}) , so we get $H^{t+1} \cap (b_t, a_{t+1}) = I_{a_{t+1}} \cap (b_t, a_{t+1})$,
- or we can't push anymore since $H^{t+1} \cap [a_{t+1}, a_1) = I_{a_{t+1}} \cap [a_{t+1}, a_1)$.

Here is an example of how the procedure works. Let us use the positroid associated to the decorated permutation in Figure 1. We will try to find a basis H that maximizes the size of the intersection with $E = [1, 2] \cup [7, 10] \cup [13, 13]$. We start with $H^1 = I_1 = \{1, 3, 4, 5, 10, 11, 12\}$. Using $I_7 = \{7, 8, 9, 10, 11, 12, 4\}$, we can push 3 and 5 (elements of $(H^1 \setminus I_7) \cap (2, 7)$) into $[7, 10]$ as 7 and 8 (elements of $(I_7 \setminus H^1) \cap [7, 10]$). We get $H^2 = \{1, 4, 7, 8, 10, 11, 12\}$. Using $I_{13} = \{13, 14, 3, 4, 5, 10, 11\}$, we can push 12 (element of $(H^2 \setminus I_{13}) \cap (10, 13)$) into $[13, 13]$ as 13 (element of $(I_{13} \setminus H^2) \cap [13, 13]$). We end up with $H = H^3 = \{1, 4, 7, 8, 10, 11, 13\}$. It is easy to check that $|H \cap E| = rk(E)$, since any basis B of the positroid has to have at least one element in $(2, 7)$ and one element in $(10, 13)$ (from $|I_7 \cap (2, 7)| = 1$ and $|I_{13} \cap (10, 13)| = 1$). In the following theorem, we show that H gives the rank of E in general.

Theorem 6. Let $E = [a_1, b_1] \cup \dots \cup [a_k, b_k]$ be a disjoint union of k cyclic intervals, where $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ are cyclically ordered. The above procedure creates $H = H^k$ such that $|H \cap E| = nbd(E, \Pi)$ for some Π , a non-crossing partition of $[k]$. As a result, we get $rk(E) = nbd(E, \Pi)$.

Proof. We use induction on k . When $k = 1$, the statement is obvious since $|I_a \cap [a, b]| = rk([a, b]) = nbd([a, b])$. Now for the sake of induction, assume the statement is true for the disjoint union of up

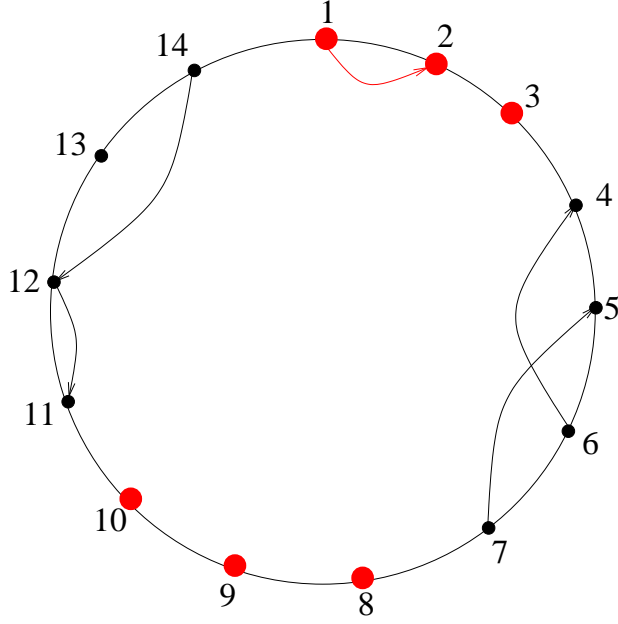


Figure 4: Information needed to compute the rank of $[1, 3] \cup [8, 10]$.

to $k - 1$ cyclic intervals. When we run the above procedure for E , if the second case never happens, then we have $H \cap (b_i, a_{i+1}) = I_{a_{i+1}} \cap (b_i, a_{i+1})$ for all $i \in [k]$. Since $I_{a_{i+1}}$ has the minimal number of possible elements a basis can have in (b_i, a_{i+1}) for each $i \in [k]$, we end up with $|H \cap E| = nbd(E)$.

Now let q be the first time when the second case happens. Partition E into $E_{head} := [a_1, b_1] \cup \dots \cup [a_q, b_q]$ and $E_{tail} := [a_{q+1}, b_{q+1}] \cup \dots \cup [a_k, b_k]$. Since we have $H^{q+1} \cap [a_{q+1}, a_1] = I_{a_{q+1}} \cap [a_{q+1}, a_1]$, notice that $H \cap E_{tail}$ is the same as $H_{tail} \cap E_{tail}$, where H_{tail} denotes what we get when we run the procedure on E_{tail} starting from $I_{a_{q+1}}$ (Although $H \cap [a_{q+1}, a_1]$ and $H_{tail} \cap [a_{q+1}, a_1]$ might differ, their intersection with E_{tail} is the same). By the induction hypothesis, we get $|H_{tail} \cap E_{tail}| = nbd(E_{tail}, \Pi_{tail})$ for some Π_{tail} , a non-crossing partition of $[q + 1, k]$. Also for the head part, we have $H \cap [a_1, a_{q+1}] = H_{head} \cap [a_1, a_{q+1}]$ where H_{head} denotes what we get when we run the procedure on E_{head} starting from I_{a_1} . Again by the induction hypothesis, we get $|H_{head} \cap E_{head}| = nbd(E_{head}, \Pi_{head})$ for some Π_{head} , a non-crossing partition of $[1, q]$. Now merge the parts of Π_{head} and Π_{tail} together to form a non-crossing partition Π of $[k]$. We get $|H \cap E| = |H_{head} \cap E_{head}| + |H_{tail} \cap E_{tail}| = nbd(E_{head}, \Pi_{head}) + nbd(E_{tail}, \Pi_{tail}) = nbd(E, \Pi)$. Since $rk(E)$ is bounded above by $nbd(E, \Pi)$, there existing a basis H with $|H \cap E| = nbd(E, \Pi)$ tells us that $rk(E) = nbd(E, \Pi)$. \square

For example, take a look at Figure 4. The rank of $E = [1, 3] \cup [8, 10]$ is bounded above by $nbd(E, \{\{1\}, \{2\}\})$ and $nbd(E, \{\{1, 2\}\})$. We get $nbd(E, \{\{1\}, \{2\}\}) = rk([1, 3]) + rk(8, 10) = 2 + 3 = 5$, since rank of an interval $[a, b]$ is given by $|[a, b]|$ minus the number of intervals of form $[\pi^{-1}(x), x]$ contained in $[a, b]$ (from I_a being given by a -exceedances, and $rk([a, b]) = |I_a \cap [a, b]|$). We also have $nbd(E, \{\{1, 2\}\}) = rk(\mathcal{M}) - minelts((3, 8)) - minelts((10, 1)) = 7 - 2 - 2 = 3$, since $minelts((b, a))$ is given by the number of intervals of form $[x, \pi^{-1}(x)]$ contained in (b, a) . Hence the above theorem tells us that $rk(E) = 3$.

6 Inseparable flats

From Theorem 6, if E is inseparable, we have $rk(E) = nbd(E)$. In this case, for any basis $B \in \mathcal{M}$ that maximizes $|B \cap E|$, the number of elements in the intervals of the complement of E , the (b_i, a_{i+1}) 's, has to be minimal.

Theorem 7. *Let E be an inseparable set. Then E is a flat if and only if the complement of E is covered by intervals of form $[x, \pi^{-1}(x)]$ that does not intersect E .*

Proof. Let $E = [a_1, b_1] \cup \dots \cup [a_k, b_k]$, where $[a_i, b_i]$'s are pairwise disjoint cyclic intervals. By Theorem 6, E being inseparable means that any basis B that maximizes $|E \cap B|$ has minimal number of possible elements in each cyclic interval of the complement. Therefore E is a flat if and only if for any $y \in E^c$, we have $B \in \mathcal{M}$ with $y \in B$ and $|B \cap (b_i, a_{i+1})| = |I_{a_{i+1}} \cap (b_i, a_{i+1})|$ for all i . Now we will replicate what we did in Section 4. Using the interval exchange property for positroids (Proposition 1) and the dual basis exchange property, the existence of $B \in \mathcal{M}$ with $y \in B$ and $|B \cap (b_i, a_{i+1})| = |I_{a_{i+1}} \cap (b_i, a_{i+1})|$ for all i is equivalent to the existence of $B' \in \mathcal{M}$ with:

- $B' \cap (b_i, a_{i+1}) = I_{a_{i+1}} \cap (b_i, a_{i+1})$ for all i except for some j ,
- and $B' \cap (b_j, a_{j+1}) = I_{a_{j+1}} \cap (b_j, a_{j+1}) \setminus \{x\} \cup \{y\}$ for that j , where $x \leq_{b_j} y$.

Now let us try to show that the existence of B' described above is equivalent to y being covered by some interval $[z, \pi^{-1}(z)]$ in (b_j, a_{j+1}) . Let us start with the direction where we have y being covered by some interval $[z, \pi^{-1}(z)]$ and try to show the existence of B' described above. If $y \in I_{a_{j+1}}$, we can just set $x = y$ and be done. So assume that $y \notin I_{a_{j+1}}$. Since y is covered by an interval of form $[z, \pi^{-1}(z)]$, in (b_j, a_{j+1}) , we have $z \in I_{a_j} \setminus I_y$. By the sharing property, we get $I_y \cap [a_j, y) \subset I_{a_j}$. Since our E is inseparable, the H obtained by the procedure starting from I_{a_j} would satisfy the following:

- $H \cap (b_i, a_{i+1}) = I_{a_{i+1}} \cap (b_i, a_{i+1})$ for all i ,
- and $H \cap [a_i, b_i] \subset I_{a_i} \cap [a_i, b_i]$, where it is an equality for $i = j$.

Using Lemma 6 with $b = b_j, c = y, a = a_{j+1}$ on H , we get $B' := H \setminus \{z\} \cup \{y\} \in \mathcal{M}$ with $z <_{b_j} y$. This B' is the basis we wanted to show the existence of.

For the other direction, assume we have $B' \in \mathcal{M}$ with the above conditions. If all of $I_{a_{j+1}} \cap (b_j, y)$ is contained in I_y , then B' would have less than minimal number of possible elements in (b_j, y) and we would get a contradiction. So there is some element $x' \in (I_{a_{j+1}} \setminus I_y) \cap (b_j, y)$. We get $\pi^{-1}(x') >_{x'} y$ from $x' \notin I_y$ and $\pi^{-1}(x') <_{x'} a_{j+1}$ from $x' \in I_{a_{j+1}}$. Hence y is covered by an interval $[x', \pi^{-1}(x')]$ contained in (b_j, a_{j+1}) . This concludes our proof. \square

By combining Theorem 7 with Corollary 2, we get the following result:

Corollary 3. *Let F be an inseparable flat. Then F is the intersection of interval flats. Let \mathcal{E} be the collection of interval flats and their intersections. Then the independent set polytope of a positroid is given by inequalities of form $x_F \leq rk(F)$ for all $F \in \mathcal{E}$.*

For example, for the positroid coming from the decorated permutation of Figure 1, one of the facets is given by $x_1 + x_2 + x_3 + x_8 + x_9 + x_{10} \leq 3$, since $rk([1, 3] \cup [8, 10]) = 3$ as we have seen in the analysis of Figure 4, following Theorem 6.

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